ON CERTAIN LINEAR DIFFERENTIAL EQUATIONS OF THE SECOND ORDER.

By F. H. MURRAY.

Differential equations of the second order and of the form

$$\frac{d^2x}{dt^2} - \alpha^2 x = \mu f(x, t, \mu)$$

have been studied by Poincaré,* Bohl,† and others.

It is the purpose of this paper to generalize certain of the results obtained by Bohl for linear equations of the second order; at the same time a simplified treatment is made possible by methods of successive approximation. Among other results an analytic expression for the general integral is obtained, from which the characteristic properties of the integral curves can be immediately deduced.

While some of the results of §§ 1 and 2 are known,‡ these have been given because of their importance for later developments.§

1. In the differential equation

$$\frac{d^2x}{dt^2} - \varphi(t)x = 0 \tag{1}$$

suppose $\varphi(t)$ continuous, and positive or zero in the interval $0 \leqq t \leqq T$. There will be no loss of generality in restricting the analytical developments which follow to the upper right-hand quadrant of the (x, t) plane, since the transformations $x' = \pm x$, $t' = \pm t$ leave (1) unchanged or replace this equation by another of the same form.

In the interval I: $0 \leqq t \leqq T$ suppose

$$0 \leqq \varphi(t) < B. \tag{2}$$

* Les Nouvelles Méthodes de la Mécanique Céleste, tome II, p. 311.

† Bulletin de la Société Mathématique de France, tome 38 (1910); see also Crelle's Journal, Band 131.

‡ See Wiman, Arkiv för Mat., Astr. och Fysik, vol. 12, Nr. 14 (1917).

§ In what follows it will be understood that discontinuous solutions are excluded from consideration.

Instead of (1), consider the equation

$$\frac{d^2 x}{d t^2} = \mu \varphi(t) x \tag{3}$$

and assume a development of the form

$$x = x_0 + \mu x_1 + \mu^2 x_2 + \cdots.$$

Suppose $x(0) = a$, $x'(0) = b$ for all values of μ.

Substituting the formal development in equation (3), we obtain the equations and boundary conditions

$$\begin{aligned} &\frac{d^2 x_0}{d t^2} = 0, \qquad x_0 = a, \qquad x_0' = b \text{ if } t = 0, \\ &\frac{d^2 x_n}{d t^2} = \varphi(t) x_{n-1}, \quad x_n = 0, \quad x_n' = 0 \text{ if } t = 0. \qquad (n = 1, 2, 3, \cdots) \end{aligned} \tag{4}$$

Consequently

$$\begin{aligned} x_0 &= a + b t \\ x_n &= \int_0^t \int_0^z \varphi(u) x_{n-1}(u)\, du\, dz, \qquad (n = 1, 2, 3, \cdots) \end{aligned} \tag{5}$$

Suppose $|x_0| < A$ in the interval I.

Then

$$|x_1| < A \int_0^t \int_0^z B\, du\, dz = A B \frac{t^2}{2},$$

and in general

$$|x_n| < A B^n \frac{t^{2n}}{(2n)!}.$$

Consequently

$$x = \sum_{n=0}^{\infty} x_n \mu^n \ll A \sum_{n=0}^{\infty} (\mu B)^n \frac{t^{2n}}{(2n)!}. \qquad (\arg \mu) \tag{6}$$

It is seen immediately that for any value of μ, the series on the left of (6) can be differentiated twice term by term, and the convergence of the differentiated series is uniform with respect to t, in the interval $0 \leqq t \leqq T$.

Consequently the function

$$x = \sum_{n=0}^{\infty} x_n \mu^n$$

satisfies equation (3); taking $\mu = 1$ we obtain the solution of (1).

From (2), (5) it follows that so long as $a+bt>0$, the functions $x_1(t)$, $x_2(t)\cdots\cdots$ are positive or zero; consequently so long as $x(t)>0$, on the interval I we shall have

$$x(t)\geqq a+bt.$$

Consider a second equation

$$\frac{d^2\bar{x}}{dt^2}-\varphi_1(t)\bar{x}=0, \tag{7}$$

with

$$\varphi_1(t)>\varphi(t), \qquad 0\leqq t\leqq T.$$

Suppose $\bar{x}(t)$ a solution of (7) satisfying the same initial conditions, and form the corresponding functions

$$\begin{aligned}&\bar{x}^0=a+bt\\ &\bar{x}_n=\int_0^t\int_0^z\varphi_1(u)\,\bar{x}_{n-1}(u)\,du\,dz, \qquad (n=1,2,\cdots\cdots)\end{aligned} \tag{8}$$

Since for all points of the interval $\varphi_1(u)>\varphi(u)$, we shall obtain

$$\bar{x}_1>x_1,\ \cdots\cdots\ \bar{x}_n>x_n \tag{9}$$

since if $\bar{x}_{n-1}>x_{n-1}$, we obtain from (8) $\bar{x}_n>x_n$; applying the method of induction we obtain the inequalities (9) for all values of n under the hypothesis $a+bt>0$.

From the equations

$$\begin{aligned}x&=x_0+x_1+x_2+\cdots\cdots\\ \bar{x}&=\bar{x}_0+\bar{x}_1+\bar{x}_2+\cdots\cdots\end{aligned}$$

it follows immediately that, so long as $a+bt>0_n$ on the interval $0<t\leqq T$, we shall have $\bar{x}(t)>x(t)$.

Another result to be obtained from the form of the expressions (5) is that since x_0 is linear in a, b, the same is true of x_n $(n=1,2,3,\cdots\cdots)$, and

$$x=aX_1+bX_2,$$

where $X_1(t)$, $X_2(t)$ are positive or zero in the interval I. These are the *principal* functions for the value $t=0$, since X_1 is obtained by taking $a=1$, $b=0$; X_2 by taking $a=0$, $b=1$.

We are now in a position to prove omit period.

THEOREM 1. *Assume $\varphi(t)$, $\varphi_1(t)$ continuous functions of t satisfying the conditions*

$$\varphi_1(t) > \varphi(t) \geqq 0, \qquad 0 \leqq t \leqq T,$$

and suppose $x(t)$, $\bar{x}(t)$ solutions of the equations and boundary conditions

$$(\alpha) \qquad \begin{aligned} &\frac{d^2 x}{d t^2} - \varphi(t)\, x = 0, \qquad x = a, \qquad x' = b \text{ if } t = 0, \\ &\frac{d^2 \bar{x}}{d t^2} - \varphi_1(t)\, \bar{x} = 0, \qquad \bar{x} = a, \qquad \bar{x}' = b \text{ if } t = 0, \end{aligned}$$

in which $a > 0$; then, if $x(t) > 0$ in an interval $0 < t < t_1 < T$, $\bar{x} > x$ in the same interval.

For from equations (α) and the condition $\bar{x}'(0) = x'(0)$,

$$(10) \qquad \bar{x}'(t) - x'(t) = \int_0^t \varphi_1(\bar{x} - x)\, dt + \int_0^t x(\varphi_1 - \varphi)\, dt.$$

Since $a > 0$ the right-hand member is positive in a certain interval for which $\bar{x} > x > 0$, as was shown above; hence in a certain interval for which $\bar{x} > x > 0$, $\frac{d}{dt}(\bar{x}' - x) > 0$. But $\bar{x} - x$ can vanish a first time at $t = t_1$, only if for some preceding value $t' < t_1$, the derivative $\frac{d}{dt}(\bar{x} - x)$ vanishes. This is impossible from (10), hence the theorem.

With the aid of the principal solutions we obtain easily

THEOREM 2. *If $\varphi(t)$ is continuous, and positive or zero on the interval $0 \leqq t \leqq T$, then if $x(t)$, $\bar{x}(t)$ are two solutions of the differential equation*

$$\frac{d^2 x}{d t^2} - \varphi(t)\, x = 0$$

which satisfy initial conditions of the form

$$\begin{aligned} x(0) &= a, & \bar{x}(0) &= \bar{a} \geqq a, \\ x'(0) &= b, & \bar{x}'(0) &= \bar{b} > b, \end{aligned}$$

we shall have $\bar{x} > x$ on the interval $0 < t \leqq T$.

For in terms of the principal functions

$$x = a X_1 + b X_2, \qquad \bar{x} = \bar{a} X_1 + \bar{b} X_2,$$
$$\bar{x} - x = (\bar{a} - a) X_1 + (\bar{b} - b) X_2.$$

Since

$$X_1 > 0, \qquad X_2 > 0, \qquad 0 < t \leqq T,$$

the theorem follows.

2. In the theorems above only finite intervals are considered, but the extension to infinite intervals is immediate; if φ_1, φ satisfy the condition

$$\varphi_1(t) > \varphi(t) \geqq 0$$

for all finite values of t, the conclusions of Theorems 1, 2 are valid if T has an arbitrary positive value.

For certain applications to be made it will be convenient to have

THEOREM 3. *If $\varphi(t)$ is continuous, and positive or zero for all finite values of t without vanishing identically, then the only solution of the differential equation*

$$\frac{d^2 x}{dt^2} - \varphi(t)\, x = 0$$

which is bounded for all values of t is the solution $x \equiv 0$.

For a value $\bar{t}$ for which $\varphi(\bar{t}) > 0$ lies within a certain interval δ: $t_1 \leqq t \leqq t_2$ for which, if α is properly chosen,

$$\varphi(t) \geqq \alpha > 0. \tag{11}$$

If $x(t)$ is identically zero in δ, x must vanish for all values of t; excluding this hypothesis we can determine a sub-interval δ': $t_1' \leqq t \leqq t_2'$ for which, if β is properly chosen,

$$\pm\, x(t) \geqq \beta > 0. \tag{12}$$

Since x and $-x$ are bounded or unbounded simultaneously we may choose the $+$ sign in (12); with this choice

$$\begin{aligned} &\varphi(t) \geqq \alpha > 0 \\ &x(t) \geqq \beta > 0 \qquad t_1' \leqq t \leqq t_2'. \end{aligned}$$

Suppose $x'(t_1') < 0$; from the developments of the first paragraph it follows that, if $t < t_1'$,

$$x(t) \geqq x(t') + x'(t_1')\,(t - t_1').$$

Consequently x can have no upper bound.

Suppose $x'(t_1') \geqq 0$; integrating the differential equation we obtain:

$$x'(t_2') - x'(t_1') = \int_{t_1'}^{t_2'} \varphi(t)\, x(t)\, dt$$
$$\geqq \alpha\beta(t_2' - t_1').$$

Consequently $x'(t_2')$ must be positive, and since

$$x(t) \geqq x(t_2') + x'(t_2')\,(t - t_2')$$

for all finite values of $t > t_2'$, x must take on values arbitrarily large in this case also; hence the theorem.

In addition to the principal solutions we shall introduce another class of particular solutions by means of.

THEOREM 4. *If $\varphi(t)$ is continuous, and positive or zero without vanishing identically for all finite values of t, then if (x_0, t_0) is an arbitrary set of initial values such that $x_0 \neq 0$, there exist two distinct solutions Y_1, Y_2 of the equation*

$$\frac{d^2x}{dt^2} - \varphi(t)\, x = 0$$

satisfying the conditions

$$Y_1(t_0) = Y_2(t_0) = x_0,$$
$$|Y_1(t)| \leqq |x_0|, \qquad t > t_0,$$
$$|Y_2(t)| \leqq |x_0|, \qquad t < t_0.$$

By means of a transformation of the form

$$x' = \pm x,\ t' = \pm(t - t_0),$$

the discussion of the general case can be reduced to that in which $x_0 > 0$, $t_0 = 0$, $t > 0$; the transformed function $\varphi(t')$ will continue to satisfy the conditions assumed.

Suppose $T > 0$, and determine $x(t)$ by the conditions

$$x(0) = x_0,\ x(T) = 0.$$

Since by definition the principal solutions satisfy the relations

$$X_1(0) = 1, \qquad X_1'(0) = 0,$$
$$X_2(0) = 0, \qquad X_2'(0) = 1,$$

and

$$X_1 X_2' - X_2 X_1' = C,$$

C can be determined by putting $t = 0$; hence $C = 1$.

Since

$$x = a X_1 + b X_2,$$
$$a X_1(0) + b X_2(0) = x_0,$$
$$a X_1(T) + b X_2(T) = 0,$$
$$a = x_0,$$
$$b = -x_0 \frac{X_1(T)}{X_2(T)}. \qquad (X_2(T) > 0)$$

Consider the function

$$z = -\frac{X_1(t)}{X_2(t)},$$

(13)
$$\frac{dz}{dt} = \frac{X_1 X_2' - X_2 X_1'}{X_2^2} = \frac{1}{X_2^2},$$

$$z(t) - z(t_1) = \int_{t_1}^{t} \frac{dt}{X_2^2}, \qquad t_1 > 0.$$

Since $X_2(0) = 0$, $X_2'(0) = 1$, it follows that, if $t > 0$, $X_2(t) \geqq t$.

Consequently

$$0 < z(t) - z(t_1) \leqq \int_{t_1}^{t} \frac{dt}{t^2}, \qquad t > t_1,$$
$$\leqq \frac{1}{t_1} - \frac{1}{t} < \frac{1}{t_1},$$

from which

(14)
$$z(t_1) < z(t) < z(t_1) + \frac{1}{t_1}, \qquad t > t_1 > 0.$$

From (13), (14) it is seen that $z(t)$ is an increasing function of t which remains less than a function of t_1; consequently z approaches a limit as t becomes infinite,

$$\lim_{t=\infty} z(t) = -\lambda.$$

Since $z < 0$, $\lambda \geqq 0$.

Suppose

$$Y_1 = x_0[X_1 - \lambda X_2], \qquad \bar{x} = x_0[X_1 + z(T) X_2].$$

If $t > 0$, consider the function $\bar{x}$ for $T > \bar{t}$,

$$\bar{x}(\bar{t}) = x_0[X_1(\bar{t}) + z(T)X_2(\bar{t})].$$

If $t < T$, $\bar{x}'(t) < 0$; for if $\bar{x}'(t_1) \geqq 0$, $t_1 < T$, $\bar{x}(T) \geqq \bar{x}(t_1)$. Hence for all values of $t \leqq T$,

$$\bar{x}(t) < x_0,$$

consequently

$$\bar{x}(\bar{t}) < x_0.$$

When T becomes infinite, z approaches a finite limit $-\lambda$; hence

$$\lim_{T=\infty} \bar{x}(\bar{t}) = x_0[X_1(\bar{t}) - \lambda X_2(\bar{t})].$$

Since $\bar{x}(\bar{t}) < x_0$ for all finite values of T,

$$\lim_{T=\infty} \bar{x}(\bar{t}) = Y_1(\bar{t}) \leqq x_0.$$

For negative values of t, z is positive and decreasing as $t \to -\infty$. Hence we may write

$$\bar{\lambda} = \lim_{t \to -\infty} z(t), \qquad \bar{\lambda} \geqq 0,$$

$$Y_2 = x_0[x_1 + \bar{\lambda} X_2].$$

A discussion similar to that already given shows that, for $t < 0$, $0 < Y_2 \leqq x_0$.

The functions $Y_1(t)$, $Y_2(t)$ satisfy the conditions of the theorem if $x_0 > 0$; for if $Y_1 \equiv Y_2$, the function Y_1 would be bounded for all values of t, contrary to Theorem 3. The corresponding functions for $x_0 < 0$ are seen to be

$$Y_1 = x_0[X_1 - \lambda X_2], \qquad Y_2 = x_0[X_1 + \bar{\lambda} X_2].$$

From these results it follows that there are three classes of integral curves through an arbitrary point not on the t-axis. Consider a point (x_0, t_0) in the upper half-plane, and an integral curve through this point with the tangent x_0'. If $x_0' < -\lambda x_0$, the function $x(t)$ passes from $+\infty$ to $-\infty$ when t passes from $-\infty$ to $+\infty$. If $-\lambda x_0 < x_0' < \bar{\lambda} x_0$, x passes from $+\infty$ to a certain minimum value $x_1 > 0$, and again becomes positively infinite when t passes from $-\infty$ to $+\infty$. If $x_0' > \bar{\lambda} x_0$, x passes from $-\infty$ to $+\infty$ when t passes from $-\infty$ to $+\infty$.

Since not all solutions remain bounded for $t > 0$, any two such solutions must satisfy a relation $\overline{Y}_1 = C Y_1$; similarly for $t < t_0$, $\overline{Y}_2 = C Y_2$.

The quantities $\lambda, \overline{\lambda}$ corresponding to a value $t = t_0$ have been defined by means of the principal solutions $X_1(t), X_2(t)$ corresponding to $t = t_0$;

$$
\begin{aligned}
\lambda(t_0) &= \lim_{t \to \infty} \frac{X_1(t)}{X_2(t)}, \\
-\overline{\lambda}(t_0) &= \lim_{t \to -\infty} \frac{X_1(t)}{X_2(t)}.
\end{aligned}
\tag{15}
$$

It is easily seen that if $\lambda(t_1) = 0$, then $\varphi(t) \equiv 0$ for $t > t_1$; if $\overline{\lambda}(t_1) = 0$, $\varphi(t) \equiv 0$ for $t < t_1$. Consequently $\lambda + \overline{\lambda} > 0$ if $\varphi(t)$ is not identically zero.

Consider the functions

$$
Y_1 = x_0[X_1 - \lambda X_2], \qquad Y_2 = x_0[X_1 + \overline{\lambda} X_2].
$$

For the value $t = t_0$,

$$
\frac{Y_1'}{Y_1} = -\lambda(t_0), \qquad \frac{Y_2'}{Y_2} = \overline{\lambda}(t_0).
\tag{16}
$$

Corresponding to each point (t_1, x_1) of the curve $x = Y_1(t)$ the solution of (1) which remains bounded for $t > t_1$ must coincide with the solution $x = Y_1(t)$; hence for every value $t_1 > t_0$, we have the relations

$$
\frac{Y_1'}{Y_1} = -\lambda(t_1), \qquad \frac{Y_2'}{Y_2} = \overline{\lambda}(t_1).
$$

Since t_1 is arbitrary, these relations may be written

$$
\frac{Y_1'}{Y_1} = -\lambda(t), \qquad \frac{Y_2'}{Y_2} = \overline{\lambda}(t),
$$

from which

$$
Y_1 = Y_1^0 e^{-\int_{t_0}^{t} \lambda\, dt}, \qquad Y_2 = Y_2^0 e^{\int_{t_0}^{t} \overline{\lambda}\, dt}
\tag{17}
$$

Since the functions Y_1, Y_2 are linearly independant, it follows that *the general solution of the differential equation* (1) *can be given the form*

$$
x = C_1 e^{-\int_{t_0}^{t} \lambda\, dt} + C_2 e^{\int_{t_0}^{t} \overline{\lambda}\, dt},
\tag{18}
$$

in which $\lambda(t)$, $\overline{\lambda}(t)$ are positive or zero.

From (18) we deduce immediately

THEOREM 5. *A necessary and sufficient condition that each solution bounded for $t > t_0$ approach a limit different from zero as $t \to \infty$ is that a constant G exist such that*

$$\text{(a)} \qquad \int_{t_0}^{t} \lambda(t)\,dt < G, \qquad t > t_0;$$

similarly, a necessary and sufficient condition that each solution bounded for $t < t_0$ approach a limit different from zero as $t \to -\infty$ is that a constant G' exist such that

$$\int_{t}^{t_0} \bar{\lambda}\,dt < G', \qquad t < t_0.$$

For condition (a) is a necessary and sufficient condition that

$$e^{-\int_{t_0}^{t} \lambda dt}$$

approach a limit different from zero as $t \to \infty$, and any solution bounded for $t > t_0$ can be represented in the form

$$x = Ce^{-\int_{t_0}^{t} \lambda dt}.$$

The second part of the theorem can be proved in a similar manner.

Suppose that $\varphi(t)$ satisfies a condition of the form

$$\varphi(t) > \alpha^2 > 0, \qquad t > t_0.$$

Applying theorem 1 to the case $\varphi_1(t) = \alpha^2$, $t \geqq t_0$, we obtain $(\varphi(t) > \varphi_1(t))$

$$x(t) > \tfrac{1}{2}\left(x_0 + \frac{x_0'}{\alpha}\right) e^{\alpha(t-t_0)} + \tfrac{1}{2}\left(x_0 - \frac{x_0'}{\alpha}\right) e^{-\alpha(t-t_0)}, \qquad t > t_0.$$

Now suppose $x(t)$ to be the function $Y_1(t)$ corresponding to $t = t_0$; since $x(t)$ is bounded,

$$x_0 + \frac{x_0'}{\alpha} \leqq 0, \qquad \alpha \leqq -\frac{x_0'}{x_0}.$$

Since $-\dfrac{x_0'}{x_0} = \lambda(t_0)$, from (16), $\alpha \leqq \lambda(t_0)$.

If $\varphi(t) < \beta^2$, $t \geqq t_0$, we obtain in a similar manner $\lambda(t_0) \leqq \beta$.

Hence if $\varphi(t)$ is a monotone increasing function for $t>t_0$, $\alpha(t_0)=\sqrt{\varphi(t_0)-\varepsilon}$, and $\lambda(t_0)>\sqrt{\varphi(t_0)-\varepsilon}$; similarly if $\varphi(t)$ is a monotone decreasing function for $t>t_0$, $\varphi(t)<\varphi(t_0)+\varepsilon$, and $\lambda(t_0)<\sqrt{\varphi(t_0)+\varepsilon}$, for $t>t_0$. If $\varphi(t)$ approaches a limit α^2 for $t\to\infty$, $\alpha^2-\varepsilon<\varphi(t)<\alpha^2+\varepsilon$, when $t>t_1$, hence $\sqrt{\alpha^2-\varepsilon}<\lambda<\sqrt{\alpha^2+\varepsilon}$, and λ approaches α.

From Theorem 1 we obtain easily the following result:

Given the equations

$$\frac{d^2x}{dt^2}-\varphi_1(t)\,x=0,$$

$$\frac{d^2x}{dt^2}-\varphi_2(t)\,x=0,$$

in which $\varphi_2>\varphi_1>0$, let $(\lambda_1,\bar\lambda_1)$, $(\lambda_2,\bar\lambda_2)$ be the pairs of functions corresponding to φ_1, φ_2 respectively; then for all values of t, $\lambda_2\geqq\lambda_1$, $\bar\lambda_2\geqq\bar\lambda_1$.

For the solution of the second equation satisfying the conditions $x_2(t_0)=x_1$, $x_2'(t_0)=x_1'$ can be written in the form

$$x_2=\frac{\bar\lambda_2^0x_1-x_1'}{\bar\lambda_2^0+\lambda_2^0}e^{-\int_{t_0}^{t}\lambda_2\,dt}+\frac{\lambda_2^0x_1+x_1'}{\lambda_2^0+\bar\lambda_2^0}e^{\int_{t_0}^{t}\bar\lambda_2\,dt};$$

if $-x_1'=\lambda_1^0x_1$, $x_1>0$,

$$\frac{x_2}{x_1}=\frac{\bar\lambda_2^0+\lambda_1^0}{\lambda_2^0+\bar\lambda_2^0}e^{-\int_{t_0}^{t}\lambda_2\,dt}+\frac{\lambda_2^0-\lambda_1^0}{\lambda_2^0+\bar\lambda_2^0}e^{\int_{t_0}^{t}\bar\lambda_2\,dt}.$$

Since* $x_1(t)\to 0$, and $x_1<x_2$, by Theorem 1, x_2 must remain positive; since the second integral on the right can not remain bounded, $\lambda_2^0\geqq\lambda_1^0$. Similarly if $x_1'=\bar\lambda_1x_1$,

$$\frac{x_2}{x_1}=\frac{\bar\lambda_2^0-\bar\lambda_1^0}{\lambda_2^0+\bar\lambda_2^0}e^{-\int_{t_0}^{t}\lambda_2\,dt}+\frac{\lambda_2^0+\bar\lambda_1^0}{\lambda_2^0+\bar\lambda_2^0}e^{\int_{t_0}^{t}\bar\lambda_2\,dt}.$$

Here again x_2 must remain positive, and $\bar\lambda_2^0\geqq\bar\lambda_1^0$. Since t_0 is an arbitrary value of t the result stated follows.

More definite results concerning the behavior of the solutions as $t\to\infty$ can be obtained if $\varphi(t)$ satisfies a condition of the form

$$\alpha^2t^{m_1}<\varphi(t)<\alpha^2t^{m_2},\qquad t>t_0,$$

(a) $$-2<m_1<m_2,$$

* The function $x_1(t)$ is assumed to satisfy the first differential equation above, and $x_1(t_0)=x_1$, $x_1'(t_0)=x_1'$.

or

(b) $$m_1 < m_2 < -2.$$

For if $m \neq -2$, the equation

(c) $$\frac{d^2 y}{dt^2} - \alpha^2 t^m y = 0$$

can be transformed into the equation*

(d) $$\frac{d^2\bar{y}}{dz^2} + \left[-\tfrac{1}{4} + \frac{\tfrac{1}{4} - p^2}{z^2}\right]\bar{y} = 0,$$

by means of the substitutions

(e) $$p = \frac{1}{m+2}, \qquad y = t^{-\frac{m}{4}}\bar{y}, \qquad z = \frac{4\alpha}{m+2}\, t^{\frac{m+2}{2}}.$$

If condition (a) is satisfied, $z \to \infty$, as $t \to \infty$, and equation (d) has a solution $\bar{y} = W_{0p}(z)$ which approaches zero as $z \to \infty$:

$$W_{0p}(z) = e^{-\frac{z}{2}}\left[1 + \frac{a_1}{z} + \cdots + \frac{a_n}{z^n} + \frac{R_n}{z^{n+1}}\right],$$

$$|R_n(z)| < c, \qquad z > z_0.$$

The corresponding solution of (c) can be written

$$y = t^{-\frac{m}{4}} e^{\frac{-2\alpha}{m+2} t^{\frac{m+2}{2}}}\left[1 + \bar{R}_1(t)\, t^{-\frac{m+2}{2}}\right].$$

Giving m the values m_1, m_2, we obtain the result that if $(\lambda_1, \bar{\lambda}_1)$, $(\lambda_2, \bar{\lambda}_2)$, $(\lambda, \bar{\lambda})$ are the functions corresponding to equation (c) and the equation (1) respectively,

$$\lambda_2 \geqq \lambda \geqq \lambda_1, \qquad \bar{\lambda}_2 \geqq \bar{\lambda} \geqq \bar{\lambda}_1.$$

Then if

$$y_1 = x_0 e^{-\int_{t_0}^{t} \lambda_1 dt}, \qquad x = x_0 e^{-\int_{t_0}^{t} \lambda dt}, \qquad y_2 = x_0 e^{-\int_{t_0}^{t} \lambda_2 dt},$$

$$y_2 \leqq x \leqq y_1, \qquad t > t_0.$$

* See Whittaker and Watson, Modern Analysis, Chap. XVI.

from which

$$C_2 t^{\frac{-m_2}{4}} e^{\frac{-2\alpha}{m_1+2} t^{\frac{m_2+2}{2}}} \left[1+\bar{R}_2 t^{\frac{-m_2+2}{2}}\right]$$
$$\leqq x \leqq C_1 t^{\frac{-m_1}{4}} e^{\frac{-2\alpha}{m_1+2} t^{\frac{m_1+2}{2}}} \left[1+\bar{R}_1 t^{\frac{-m_1+2}{2}}\right].$$

In case (b) $z \to 0$ as $t \to \infty$; the solutions of the transformed equation can be given in terms of the functions $M_{0p}(z)$, $M_{0,-p}(z)$ where

$$M_{0p}(z) = z^{\frac{1}{2}+p}\left[1+\sum_{n=1}^{\infty} a_n z^{2n}\right].$$

From (e),

$$y = C\left[1+\sum_{n=1}^{\infty} a'_n t^{n(m+2)}\right].$$

Consequently in case (b) the solutions of (c) which remain bounded approach limits different from zero as $t \to \infty$, and the same must be true of the corresponding solutions of (1); for, as in case (a), we obtain inequalities $y_1 \leqq x \leqq y_2$, where $y_1(t_0) = y_2(t_0) = x(t_0)$, y_1, y_2 satisfying equations of type (c), and both y_1 and y_2 approach positive limits as $t \to \infty$.

Results concerning the manner in which solutions of (1) become infinite as $t \to \infty$ can be easily obtained from the comparison theorems above in cases (a) and (b), but these will not be developed here.

Similar remarks apply to the function $\bar{\lambda}(t)$, for $t < t_0$.

3. The results of the first two paragraphs can be extended to the more general equation

$$\frac{d}{dt}\left[k(t)\frac{dx}{dt}\right] - \varphi(t)x = 0, \tag{19}$$

if $k(t)$ is continuous and satisfies a relation of the form

$$0 < a \leqq k(t) \leqq b$$

for all real, finite values of t. For this equation can be written

$$k(t)\frac{d}{dt}\left[k(t)\frac{dx}{dt}\right] - k(t)\varphi(t)x = 0. \tag{19'}$$

Suppose

$$u(t) = \int_0^t \frac{dt}{k(t)}.$$

The relation between u and t is one-to-one and continuous; if

$$\varphi(u) = k(t)\varphi(t),$$

equation (19′) can be given the form

$$\frac{d^2 x}{du^2} - \varphi(u)x = 0. \tag{20}$$

Since this equation is of the type (1), the results already obtained can be applied.

4. In this paragraph will be given a qualitative study of the solutions of the non-homogeneous equation

$$\frac{d^2 x}{dt^2} - \varphi(t)x = \psi(t) \tag{21}$$

in which φ, ψ are continuous for all real, finite values of t, and satisfy conditions of the form

$$0 < \beta^2 < \varphi(t) < \alpha^2, \qquad |\psi(t)| < A. \tag{22}$$

Equation (21) can be given the form

$$\frac{d^2 x}{dt^2} - \alpha^2 x = [\varphi(t) - \alpha^2]x + \psi(t).$$

If $f(t) = \varphi(t) - \alpha^2$, then $|f| < \alpha^2 - \beta^2 = \gamma$. Replace $f(t)$ by $\mu f(t)$:

$$\frac{d^2 x}{dt^2} - \alpha^2 x = \mu f(t)\, x + \psi(t). \tag{23}$$

Assume a development of the form

$$x = x_0 + \mu x_1 + \cdots + \mu^n x_n + \cdots. \tag{24}$$

From (23)

$$\begin{aligned} &\frac{d^2 x_0}{dt^2} - \alpha^2 x_0 = \psi(t), \\ &\frac{d^2 x_n}{dt^2} - \alpha^2 x_n = f(t)\, x_{n-1}, \qquad (n = 1, 2, 3, \cdots) \end{aligned} \tag{25}$$

The general solution of the first equation can be obtained in the form

$$x_0 = \frac{1}{2\alpha}\left[e^{\alpha t}\int_{c_1}^{t} e^{-\alpha u}\psi(u)\,du - e^{-\alpha t}\int_{c_2}^{t} e^{\alpha u}\psi(u)\,du\right].$$

Suppose $\alpha > 0$; a particular solution can be obtained by choosing $C_1 = \infty$, $C_2 = -\infty$, since the infinite integrals converge,

$$x_0 = \frac{-1}{2\alpha}\left[e^{\alpha t}\int_t^\infty e^{-\alpha u}\,\psi(u)\,du + e^{-\alpha t}\int_{-\infty}^t e^{\alpha u}\,\psi(u)\,du\right]. \tag{26}$$

From (22)

$$|x_0| < \frac{A}{2\alpha}\left[e^{\alpha t}\int_t^\infty e^{-\alpha u}\,du + e^{-\alpha t}\int_{-\infty}^t e^{\alpha u}\,du\right] = \frac{A}{\alpha^2}.$$

Similarly, we may choose

$$x_n = \frac{-1}{2\alpha}\left[e^{\alpha t}\int_t^\infty e^{-\alpha u} f(u)\,x_{n-1}(u)\,du + e^{-\alpha t}\int_{-\infty}^t e^{\alpha u} f(u)\,x_{n-1}(u)\,du\right]. \tag{27}$$

If $|x_{n-1}| < A_{n-1}$, then $|x_n| < \frac{\gamma}{\alpha^2}A_{n-1} = A_n$. Hence $|x_0| < \frac{A}{\alpha^2}$,

$$|x_n| < \frac{A\gamma^n}{\alpha^{2n+2}}, \qquad (n = 1, 2, 3, \cdots) \tag{28}$$

The series (24) is dominated by the series

$$\frac{A}{\alpha^2}\left[1 + \frac{\mu\gamma}{\alpha^2} + \left(\frac{\mu\gamma}{\alpha^2}\right)^2 + \cdots + \left(\frac{\mu\gamma}{\alpha^2}\right)^n + \cdots\right], \tag{29}$$

and must therefore converge uniformly for all real values of t when $\mu = 1$, since $\gamma < \alpha^2$. The function $x(t)$ has the upper bound

$$\frac{A}{\alpha^2}\,\frac{1}{1-\frac{\gamma}{\alpha^2}} = \frac{A}{\beta^2}.$$

The differentiated series also converges uniformly, consequently the function $x(t)$ is a particular solution of equation (21) which remains bounded for all real values of t. There can be only one such solution, since the difference of two particular solutions satisfies the homogeneous equation

$$\frac{d^2x}{dt^2} - \varphi(t)\,x = 0,$$

to which Theorem 3 can be applied. We have thus proved

THEOREM 6. *If $\varphi(t)$, $\psi(t)$ are continuous and satisfy conditions of the form*

$$0<\beta^2<\varphi(t)<\alpha^2, \quad |\psi(t)|<A,$$

for all real, finite values of t, then there is one and only one solution of the differential equation

$$\frac{d^2x}{dt^2}-\varphi(t)\,x=\psi(t),$$

which remains bounded for all real values of t.

The method of successive approximation employed in the demonstration of Theorem 6 is convenient for later developments; but with the aid of the results of § 2 it can be shown that *under the conditions* $0<\beta^2<\varphi(t)$, $|\psi(t)|<A$, *there exists just one solution of the equation*

$$\frac{d^2x}{dt^2}-\varphi(t)\,x=\psi(t),$$

which remains bounded for all real values of t.

For suppose

$$y_1=e^{-\int_{t_0}^{t}\lambda dt}, \qquad y_2=e^{\int_{t_0}^{t}\bar\lambda dt}$$

Then

$$y_1y_2'-y_2y_1'=(\lambda+\bar\lambda)\,e^{\int_{t_0}^{t}(\bar\lambda-\lambda)dt}=C,$$

$$=\lambda(t_0)+\bar\lambda(t_0).$$

Consequently

$$\text{(a)} \qquad \frac{e^{\int_{t_0}^{t}\lambda dt}}{\lambda(t)+\bar\lambda(t)}=\frac{e^{\int_{t_0}^{t}\bar\lambda dt}}{\lambda(t_0)+\bar\lambda(t_0)}.$$

If t_1, t_2 are arbitrary values of t, a solution of the non-homogeneous equation can be given the form

$$x=\frac{-1}{\lambda(t_0)+\bar\lambda(t_0)}\left[\int_{t_1}^{t}\psi(z)\,y_1(t)\,y_2(z)\,dz+\int_{t}^{t_2}\psi(z)\,y_2(t)\,y_1(z)\,dz\right]$$

$$=\frac{-1}{\lambda(t_0)+\bar\lambda(t_0)}\left[\int_{t_1}^{t}\psi(z)\,e^{-\int_{t_0}^{t}\lambda du+\int_{t_0}^{z}\bar\lambda du}dz+\int_{t}^{t_2}\psi(z)\,e^{-\int_{t_0}^{z}\lambda du+\int_{t_0}^{t}\bar\lambda du}dz\right].$$

From (a) and the corresponding equation with t replaced by z, we obtain

$$-x = \int_{t_1}^{t} \frac{\psi(z) e^{-\int_z^t \lambda du}}{\lambda(z)+\bar{\lambda}(z)} dz + \int_{t}^{t_2} \frac{\psi(z) e^{-\int_t^z \bar{\lambda} du}}{\lambda(t)+\bar{\lambda}(t)} dz.$$

Since $\varphi > \beta^2 > 0$, it follows from the results of § 2 that $\lambda \geqq \beta$, $\bar{\lambda} \geqq \beta$. Consequently

$$|x| \leqq \frac{A}{2\beta}\left[\int_{t_1}^{t} e^{-\int_z^t \beta du} dz + \int_{t}^{t_2} e^{-\int_t^z \beta du} dz\right]$$

$$\leqq \frac{A}{2\beta^2}[2 - e^{-\beta(t-t_1)} - e^{\beta(t-t_2)}].$$

Hence the integrals defining x converge if $t_1 = -\infty$, $t_2 = +\infty$, and $|x| \leqq \frac{A}{\beta_2}$. The function

$$x = -\int_{-\infty}^{t} \frac{\psi(z) e^{-\int_z^t \lambda du}}{\lambda(z)+\bar{\lambda}(z)} dz - \int_{t}^{\infty} \frac{\psi(z) e^{-\int_t^z \bar{\lambda} du}}{\lambda(t)+\bar{\lambda}(t)} dz$$

is one solution of the non-homogeneous equation which remains bounded, and by Theorem 3 there can be only one such solution.

By means of the transformation of § 3 these results can be extended to the equation

$$\frac{d}{dt}\left[k(t)\frac{dx}{dt}\right] - \varphi(t)x = \psi(t), \tag{30}$$

if (30) is written in the form

$$\frac{d^2x}{du^2} - \Phi(u)x = \Psi(u),$$

$$\Phi(u) = k(t)\varphi(t), \qquad \Psi(u) = k(t)\psi(t).$$

In particular the series expansion of which the general term is given by (27) can be employed.

From the form of (27) the sign of $x(t)$ is constant if $\psi(t) \neq 0$. For suppose $\psi(t) \geqq 0$ without vanishing identically. Then from (26), $x_0 < 0$. Since $f(t)$ is negative, x_n has the sign of x_{n-1}, hence $x_1 < 0$, $x_2 < 0$, $\cdots\cdots$; consequently $x < 0$ for all real values of t. Similarly if $\psi \leqq 0$, $x > 0$.

5. Differential equations of the type considered here are especially important when the coefficients φ, ψ are quasi-periodic* functions of t.

By definition $f(t)$ is a uniformly continuous quasi-periodic function of t with the periods $\alpha_1, \alpha_2, \cdots \alpha_m$ if, given $\varepsilon > 0$, an η can be determined such that

$$|f(t+\tau)-f(t)| < \varepsilon$$

for all real values of t when τ satisfies the conditions

$$\left|\frac{\tau}{\alpha_i} - n_i\right| < \eta, \qquad (i = 1, 2, \cdots m)$$

$n_1 \cdots n_m$ being integers. In the works of Bohl it is shown that a function satisfying these conditions is necessarily bounded.

It is proposed to demonstrate the following

THEOREM 7. *If $k(t)$, $\varphi(t)$, $\psi(t)$ are uniformly continuous quasi periodic functions of t with the periods $\alpha_1 \cdots \alpha_m$, and such that*

$$0 < b^2 < k(t) < a^2, \qquad 0 < \beta^2 < \varphi(t) < \bar{\alpha}^2, \qquad |\psi(t)| < A$$

$$\int \frac{dt}{k(t)} = ct + g(t),$$

$g(t)$ being a uniformly continuous quasi-periodic function, then there exists just one solution of the differential equation

$$\frac{d}{dt}\left[k(t)\frac{dx}{dt}\right] - \varphi(t)x = \psi(t),$$

which is bounded for all real values of t; this solution is quasi-periodic with the periods $\alpha_1, \alpha_2 \cdots \alpha_m$.

It was shown by Bohl that $g(t)$ is quasi-periodic with the same periods $\alpha_1 \cdots \alpha_m$.

Taking

$$z = \int_0^t \frac{dt}{k(t)} = ct + g(t), \qquad c > 0,$$

$$\Phi(z) = k(t)\,\varphi(t), \qquad \Psi(z) = k(t)\,\psi(t),$$

* See Esclangon, Nouvelles Recherches sur les fonctions quasi-periodiques, Annales de l'Observatoire de Bordeaux, XVI, (1917).

we obtain the differential equation

$$\frac{d^2x}{dz^2} - \Phi(z)x = \Psi(z). \tag{31}$$

It has already been seen that there exists just one bounded solution of this equation; a uniformly convergent series expansion of this solution can be obtained from (26), (27), replacing t by z. It will be shown that each term of this series is quasi-periodic with the periods $\alpha_1 \cdots \alpha_m$.

From (26)

$$x_0 = \frac{-1}{2\alpha}\left[\int_z^\infty e^{-\alpha(u-z)}\Psi(u)\,du + \int_{-\infty}^z e^{-\alpha(z-u)}\Psi(u)\,du\right].$$

Substituting

$$u = \int_0^v \frac{dv}{k(v)},$$

then

$$x_0 = \frac{-1}{2\alpha}(I_1 + I_2),$$

where

$$I_1 = \int_z^\infty e^{-\alpha(u-z)}\Psi(u)\,du = e^{\alpha g(t)}\int_t^\infty e^{-\alpha c(v-t)}[e^{-\alpha g(v)}\psi(v)]\,dv.$$

If $g(t)$ is quasi-periodic, $e^{-\alpha g(t)}$ is quasi-periodic with the same periods; since the product of two quasi-periodic functions is quasi-periodic, it will be sufficient to show that, if $P(t)$ is a uniformly continuous quasi-periodic function with the periods $\alpha_1 \cdots \alpha_m$, the same is true of

$$Q(t) = \int_t^\infty e^{-\alpha c(v-t)}P(v)\,dv.$$

In the integral,

$$Q(t+\tau) = \int_{t+\tau}^\infty e^{-\alpha c(v-t-\tau)}P(v)\,dv,$$

substitute $v = w + \tau$:

$$Q(t+\tau) = \int_t^\infty e^{-\alpha c(w-t)}P(w+\tau)\,dw,$$

$$\begin{aligned} Q(t+\tau) - Q(t) &= \int_t^\infty e^{-\alpha c(w-t)}[P(w+\tau) - P(w)]\,dw \\ &= [P(\zeta+\tau) - P(\zeta)]\int_t^\infty e^{-\alpha c(w-t)}\,dw \\ &= \frac{1}{\alpha c}[P(\zeta+\tau) - P(\zeta)]. \qquad (\zeta > t) \end{aligned}$$

Hence if τ is so chosen that for all values of t, $|P(t+\tau)-P(t)|<\varepsilon$, then $|Q(t+\tau)-Q(t)|<\dfrac{\varepsilon}{\alpha c}$.

It follows that $Q(t)$ is a uniformly continuous quasi-periodic function with the same periods, and the same is true of $I_1(t)$. An almost identical discussion shows that $I_2(t)$ is likewise quasi-periodic.

Under the assumption that $x_{n-1}(t)$ and $\bar{f}(t)$ are uniformly continuous quasi-periodic functions with the given periods, ($\bar{f}(t)$ corresponding to $f(t)$ in equation (27)), the same discussion shows that $x_n(t)$ satisfies these conditions also. Hence by the principle of induction each term of the uniformly convergent series

$$S(t)=x_0+x_1+\cdots+x_n+\cdots$$

satisfies these conditions, from which it follows immediately that the function $S(t)$ is a uniformly continuous quasi-periodic function with the periods $\alpha_1\cdots\alpha_m$.

Making use of the fact that $k(t)>0$, we obtain the result that *the general solution of the equation*

$$\frac{d}{dt}\left[k(t)\frac{dx}{dt}\right]-\varphi(t)\,x=\psi(t)$$

can be given the form

$$x=C_1\,e^{-\int\lambda\,dt}+C_2\,e^{\int\bar{\lambda}\,dt}+S(t),$$

in which $\lambda(t)$, $\bar{\lambda}(t)$ *are positive or zero, and* $S(t)$ *is quasi-periodic with the periods* $\alpha_1\cdots\alpha_m$.